Exercise Set Solutions #2

"Discrete Mathematics" (2025)

E1. **(a)** Prove that $\binom{n}{k} \leq \binom{n}{k+1}$ when $1 \leq k < \lceil n/2 \rceil$.

> ***Solution:*** We simply write what we wish to prove as
>
> $$\binom{n}{k} = \frac{n!}{k!(n-k)!} \leq \binom{n}{k+1} = \frac{n!}{(k+1)!(n-k-1)!}$$
>
> This is equivalent to
>
> $$k+1 \leq n-k \iff 2k+1 \leq n.$$
>
> This holds exactly when $1 \leq k < \lceil n/2 \rceil$, as desired.

(b) For which values of n and k is $\binom{n}{k+1}$ twice the previous entry in the Pascal Triangle (i.e. the entry to its left)?

> ***Solution:*** The desired property is equivalent to
>
> $$\binom{n}{k+1} = 2\binom{n}{k} \iff 2(k+1) = n-k \iff 3k+2 = n$$

E2. Prove the following equations.

(a)

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

> ***Solution:*** Note that $\binom{2n}{n}$ is the number of ways to choose n elements of the set $[2n]$. Let $k \in [n]$, then $\binom{n}{k}$ is the number of ways to choose k elements from $[n]$. Furthermore, $\binom{n}{n-k}$ is the number of possibilities to choose $n-k$ elements from $\{n+1, \ldots, 2n\}$. Hence, there are $\binom{n}{k}\binom{n}{n-k}$ ways to choose n elements from $[2n]$ such that k elements are in $[n]$ and the remaining $n-k$ elements are in $\{n+1, \ldots, 2n\}$. Moreover, $\binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2$. In order to obtain the number of possibilities to choose n elements of $[2n]$, we have to sum over $k = 0, \ldots, n$. We have shown that the number of ways to obtain n elements of $[2n]$ is equal to $\sum_{i=0}^{n} \binom{n}{i}^2$. Therefore, $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$ holds.
>
> Next we give a further way to prove the equality. For this we consider the coefficient of x^n in $(1+x)^{2n}$. Due to Theorem 1.8 in the lecture, the coefficient is equal to $\binom{2n}{n}$. Furthermore, due to Theorem 1.8 we know that $(1+x)^{2n} = ((1+x)^n)^2 = \left(\sum_{i=0}^{n} \binom{n}{i} x^i\right)^2$. Since $\binom{n}{i}\binom{n}{n-i} = \binom{n}{i}^2$, the coefficient of x^n in $\left(\sum_{i=0}^{n} \binom{n}{i} x^i\right)^2$ is equal to $\sum_{i=0}^{n} \binom{n}{i}^2$.

(b)

$$\sum_{k=q}^{n} \binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}$$

Solution: Consider the set

$$\{(A,B)|A\subseteq B\subseteq [n],|A|=q,|B|\geq q\}$$

What is the cardinality of this set? One way to see the cardinality is to write it as

$$\bigsqcup_{k=q}^{n}\{(A,B)|A\subseteq B\subseteq [n],|B|=k,|A|=q\}.$$

This gives us the left-hand side of the equation. The second way to count this is to write it as

$$\{(A,A\sqcup C)|A\subseteq [n],|A|=q,C\subseteq [n]\backslash A\}.$$

This gives us the right side of the equation since $|[n]\backslash A|=n-q\Rightarrow C$ can be chosen in 2^{n-q} ways.

E3. Show that $\left(4n^3-3n+2023\right)\left(n^6+3n^4+256\right)=O\left(n^9\right)$

Solution: Notice that for $n\geq 2023$

$$\begin{aligned}\frac{1}{n^9}|\left(4n^3-3n+2023\right)\left(n^6+3n^4+256\right)| &= |\left(4-3\frac{1}{n^2}+2023\frac{1}{n^3}\right)\left(1+3\frac{1}{n^2}+256\frac{1}{n^6}\right)| \\ &\leq (4+1+1)\cdot(1+1+1) \\ &= 18.\end{aligned}$$

Thus, $\left(4n^3-3n+2023\right)\left(n^6+3n^4+256\right)=O\left(n^9\right)$.

E4. Prove the following equations

(a) $n^a=O\left(a^n\right)$ for any $a>1$.

Solution: In order to prove $n^a=O\left(a^n\right)$ for any $a>1$, we will show that $a^n\geq n^a$ for sufficient large n. Note that $a^n\geq n^a\Leftrightarrow n\ln a\geq a\ln n$. Since $\lim_{n\to\infty}\frac{n\ln a}{a\ln n}=\infty$, $n^a=O\left(a^n\right)$ holds.

(b) $n^a=O\left(n^b\right)$ for any $a\leq b$.

Solution: $n^b=n^an^{b-a}$. Since $b-a\geq 1, n^{b-a}\geq 1$.Therefore $n^a\leq n^b$ and thus $n^a=O\left(n^b\right)$.

(c) $2^n+n^2=O\left(2^n\right)$.

Solution: Note that $2^n+n^2\leq 2\cdot 2^n$ for all $n\geq 4$. Therefore $2^n+n^2=O\left(2^n\right)$.

(d) $\frac{n}{\log n}=o(n)$.

Solution: $\lim_{n\to\infty}\frac{\frac{n}{\log n}}{n}=\lim_{n\to\infty}\frac{1}{\log n}=0$, implies $\frac{n}{\log n}=o(n)$.

E5. Suppose there are 20 students and each student has to choose a number in [100]. What are the chances that all students choose a different number? If the number of students is increased, does this probability increase or decrease? How many students should there be so that the probability that two students have chosen the same number is at least 50% ?

> ***Solution:*** Due to Proposition 1.7 of the lecture, the probabilty that all 20 students choose a different number in [100] is $\frac{100!/80!}{100^{20}} = 0.130399501820471$, hence the probablity is around 13.04%. When the number of students increase, the coincidence that two students choose the same number would also increase and therefore the probability that they all choose different numbers would decrease. In general, the probabilty that n students each choose a number in [100] without two people choosing the same number is
>
> $$p_n = \frac{100 \cdot 99 \cdot 98 \cdot \dots \cdot (100-n)}{100^n}$$
>
> Clearly $p_n = \left(\frac{100-n}{100}\right) p_{n-1} < p_n$. With this, the chance that 2 out of 12 students share the same number in [100] is $1 - \frac{100!}{88!100^{12}} \approx 0.4968466$. Whereas for 13 students the probability is $1 - \frac{100!}{87!100^{13}} \approx 0.55722$. Hence, if there are at least 13 students the chances that two of them share the same number is at least 50%.

E6. Use Stirling's formula to estimate $1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)$.

> ***Solution:*** Note that
>
> $$1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) = \frac{(2n)!}{2 \cdot 4 \cdot 6 \dots \cdot 2n} = \frac{(2n)!}{2^n n!}$$
>
> Now we can use Stirlings formula, i.e. $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Note that this implies $\frac{1}{n!} \sim \frac{1}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)^n$. We obtain
>
> $$\frac{(2n)!}{2^n n!} \sim \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2^n\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = \frac{2^{1/2}\sqrt{2\pi n}2^{2n}\left(\frac{n}{e}\right)^{2n}}{2^n\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 2^{n+1/2}\left(\frac{n}{e}\right)^n$$

E7. Find a combinatorial proof of the following identity

$$\sum_{\substack{k \le n \\ k \text{ even}}} \binom{n}{k} = \sum_{\substack{k \le n \\ k \text{ odd}}} \binom{n}{k}$$

> ***Solution:*** The left-hand side above is simply the number of subsets of [n] with an even number of elements, and the right-hand side is the number of subsets with an odd number of elements. Let n then be odd. Since n is odd, selecting a set A with $|A|$ even is the same as selecting A^c with $|A^c|$ odd, and the identity is proved.
>
> Now, for n even, we observe the following: let A be a subset of $[n]$ with an even number of elements. If $n \in A$, then $A \backslash \{n\}$ is a set of odd cardinality within $[n-1]$, and if $n \notin A$, then A is a set of even cardinality within $[n-1]$. Doing the same but with sets of odd cardinality, we obtain that the number of sets A with even cardinality in $[n]$ is the same as the number of sets B with odd cardinality in $[n]$, as desired.